Exam: Solution

Exam Question 1:

Question 1.1:

1. The model has ARCH effects (ACF significant)
2. Leverage effects can be modelled
3. The ARCH(1) model is misspecified (as expected) - but only slightly so.
4. The $\hat{a} \simeq \alpha_0 + \beta_0$ in the simulations - thereby compensating and misspecification is an issue.

Question 1.2: Formally (some but not all details expected in answer):

$$\begin{aligned}
E\left(\delta\left(x_t\right)|x_{t-1}\right) &= 1 + \sigma_t^2 \\
&= 1 + \left(\omega + \alpha|x_{t-1}| + \beta|x_{t-1}|s_t\right)^2 \\
&= 1 + \omega^2 + \alpha^2 x_{t-1}^2 + \beta^2 x_{t-1}^2 s_t + c|x_{t-1}| \\
&\leq 1 + \omega^2 + \max\left(\alpha^2, \beta^2\right) x_{t-1}^2 + c|x_{t-1}| \\
&= \max\left(\alpha^2, \beta^2\right)\left(\frac{x_{t-1}^2}{\delta\left(x_{t-1}\right)}\right)\delta\left(x_{t-1}\right) + \left(\frac{1 + \omega^2 + c|x_{t-1}|}{\delta\left(x_{t-1}\right)}\right)\delta\left(x_{t-1}\right)
\end{aligned}$$

if $\max(\alpha^2, \beta^2) < 1$, then as $x_{t-1}^2 \to \infty$, $\left(\frac{1+\omega^2+c|x_{t-1}|}{\delta(x_{t-1})}\right) \to 0$ and hence

$$E\left(\delta\left(x_t\right)|x_{t-1}\right) \leq \phi\delta\left(x_{t-1}\right).$$

Question 1.3: Set

$$f_t = \left(\frac{x_t^2}{\sigma_t^2} - 1\right)\left(\frac{|x_{t-1}|s_t}{\sigma_t}\right).$$

Note that $\left(\frac{x_t^2}{\sigma_t^2} - 1\right) = \left(z_t^2 - 1\right)$ and hence,

$$\begin{aligned}
E\left(f_t \mid x_{t-1}\right) &= \left(\frac{|x_{t-1}|s_t}{\sigma_t}\right) E\left(z_t^2 - 1\right) = 0 \\
E\left(f_t^2\right) &= E\left(E\left(f_t^2|x_{t-1}\right)\right) \\
&= 2E\left(\frac{x_{t-1}^2 s_t}{\sigma_t^2}\right)
\end{aligned}$$

since $E\left(z_t^2-1\right)^2=2$. CLT:CLT in Theorem II.1 since x_t is weakly mixing (if $\alpha, \beta < 1$).

Question 1.4: The model is clearly misspecified (ARCH and Normality) - so definitely not a good model for these data. Try to find better model.

Exam Question 2:

Question 2.1: The series show serious irregularities - changing mean? changing vol? - clear ARCH effects from ACF picture. A classic GARCH cannot capture this (would expect IGARCH effects at least).

Question 2.2:

$$\hat{\mu}_1=\frac{\sum_{t=1}^{T} 1\left(s_t=1\right) x_t}{\sum_{t=1}^{T} 1\left(s_t=1\right)}$$

This is the sample mean in regime 1.

Question 2.3: Simple differentitation gives,

$$\tilde{\mu}_1=\frac{\sum_{t=1}^{T} p_t^*(1)\, x_t}{\sum_{t=1}^{T} p_t^*(1)},$$

which are the sample variance of regime one where the probabilities of being in state 1 at time t is used as weights. This is the MLE using the EM algorithm.

Question 2.4: The stationarity and ergodicity conditions:

$$p_{11}, p_{22}<1 \text{ and } p_{11}+p_{22}>0.$$

These are often (closed to at least) violated in practice as also seen in Table 2.1. Much like the IGARCH models. Clear shifts in mean and variance!